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COMMENT

Indecomposable generalisations of the standard angular momentum representations of SO(3)

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Abstract. In this comment indecomposable representations of the Lie algebra of the rotation group SO(3) are studied on the Heisenberg-Weyl basis, which is the basis for a subspace of a quotient space of the space of the universal enveloping algebra of the Heisenberg-Weyl algebra. We obtain an infinite-dimensional indecomposable representation which induces the finite-dimensional representation on certain quotient spaces and subduces the infinite-dimensional indecomposable representations on invariant subspaces. The standard angular momentum representations of SO(3) are given as special cases.

Gruber and Klimyk [1] have studied the master representation of the Lie algebra of SU(2)(SO(3)) on its universal enveloping algebra and the indecomposable representations on quotient spaces of invariant subspaces. However, we discuss an indecomposable representation of SO(3) obtained from an indecomposable representation of the two-state Heisenberg-Weyl algebra, the representation space of which is related to the universal enveloping algebra of the Heisenberg-Weyl algebra. This representation induces the finite-dimensional representations of any dimension that are completely reducible on the quotient spaces and subduces indecomposable representations that describe the transformations between quantum states with integer angular momentum and quantum states with half-integer angular momentum on the invariant subspaces. The method used to obtain these representations is analogous to that used by Gruber and Lenczewski [2] in their approach to SO(3) representations which was based on the Lorentz algebra rather than, as here, the Heisenberg-Weyl algebra.

The creation operators b_i^+, the annihilation operators b_i and the unit operator e are regarded as the generators for the N-state Heisenberg-Weyl algebra. According to the Poincaré-Birkhoff-Witt theorem, we choose for the universal enveloping algebra Ω of the Heisenberg-Weyl algebra a basis

$$f(k_i, S_i, r) = e^r \prod_{i=1}^{N} (b_i^{+k_i} b_i^{S_i}) \qquad k_i, S_i, r \in \mathbb{N}, i = 1, 2, \ldots, N \tag{1}$$

where $\mathbb{N}$ is the set of the non-negative integers and $f_{(0,0,0)} = 1$. For the left ideal L of Ω generated by the element $e-1$, the quotient space $\bar{\Omega} = \Omega/L$ is spanned by the basis

$$f(k_i, S_i) = f(k_i, S_i, 0) \bmod L. \tag{2}$$

We define this to be the Heisenberg-Weyl basis which is an extension of the Fock space of the boson system.

Extending the discussions about indecomposable representations of the one-state Heisenberg-Weyl algebra given by Gruber *et al* [3], we obtain an indecomposable representation of the N-state Heisenberg-Weyl algebra on the Heisenberg-Weyl basis

$$\begin{aligned}&\rho(b_t^+)f(k_i, S_i)=f(k_i+\delta_{it}, S_i)\\&\rho(b_t)f(k_i, S_i)=f(k_i, S_i+\delta_{it})+k_t f(k_i-\delta_{it}, S_i)\\&\rho(e)f(k_i, S_i)=f(k_i, S_i).\end{aligned}\tag{3}$$

From the above representation (3), a representation Γ of $\bar{\Omega}$ is constructed as

$$\Gamma(f(k_i, S_i))=\prod_{t=1}^{N}[\rho(b_t^+)^{k_t}\rho(b_t)^{S_t}].\tag{4}$$

There exists a subalgebra of $\bar{\Omega}$ with the elements

$$J_+=b_1^+b_2 \qquad J_-=b_2^+b_1 \qquad J_3=\tfrac{1}{2}[b_1^+b_1-b_2^+b_2]\tag{5}$$

that satisfy the closed commutation relations

$$[J_+, J_-]=2J_3 \qquad [J_3, J_\pm]=\pm J_\pm\tag{5$'$}$$

and span the Lie algebra of the three-dimensional rotation group SO(3).

On the Heisenberg-Weyl basis with the explicit form $\{f(k_1, k_2, S_1, S_2)\}$ for $N=2$, one obtains a representation of SO(3) as

$$\begin{aligned}&\rho(J_+)f(k_1, k_2, S_i, S_2)=f(k_1+1, k_2, S_1, S_2+1)+k_2f(k_1+1, k_2-1, S_1, S_2)\\&\rho(J_-)f(k_1, k_2, S_1, S_2)=f(k_1, k_2+1, S_1+1, S_2)+k_1f(k_1-1, k_2+1, S_1, S_2)\\&\rho(J_3)f(k_1, k_2, S_1, S_2)=\tfrac{1}{2}[f(k_1+1, k_2, S_1+1, S_2)\\&\qquad -f(k_1, k_2+1, S_1, S_2+1)+(k_1-k_2)f(k_1, k_2, S_1, S_2)]\end{aligned}\tag{6}$$

by using (3) and (4). It follows that the indices s_1, s_2 and the sum k_1+k_2 do not decrease under the action of ρ from equation (6) and the space $V(N, \alpha_1, \alpha_2)$

$$\{f(k_1, k_2, S_1+\alpha_1, S_2+\alpha_2)|k_1+k_2\geqslant N\in\mathbb{N}\}\tag{7}$$

for the fixed N, α_1, $\alpha_2\in\mathbb{N}$, is an SO(3) invariant subspace of $\bar{\Omega}$. On the quotient space $\bar{\Omega}/L_{(12)}$

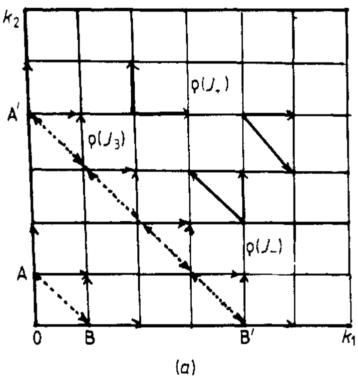
$$\{F(k_1, k_2)=f(k_1, k_2, 0, 0) \bmod L_{(12)}\}\tag{8}$$

corresponding to the left ideal $L_{(12)}$ generated by the elements $b_1-\Lambda_1$ and $b_2-\Lambda_2$ ($\Lambda_1, \Lambda_2\in$ the complex field $\mathbb{C}$), there is an induced representation

$$\begin{aligned}&\rho(J_+)F(k_1, k_2)=\Lambda_2F(k_1+1, k_2)+k_2F(k_1+1, k_2-1)\\&\rho(J_-)F(k_1, k_2)=\Lambda_1F(k_1, k_2+1)+k_1F(k_1-1; k_2+1)\\&\rho(J_3)F(k_1, k_2)=\tfrac{1}{2}[\Lambda_1F(k_1+1, k_2)-\Lambda_2F(k_1, k_2+1)+(k_1-k_2)F(k_1, k_2)].\end{aligned}\tag{9}$$

In order to study the properties of this representation, we draw the lattice diagram of representation in figure 1. Figures 1(a) and (b) correspond to the cases with Λ_1, $\Lambda_2\neq 0$ and $\Lambda_1=\Lambda_2=0$, respectively, in which each point (k_1, k_2) of the lattice denotes a basis vector $F(k_1, k_2)$ and each arrow the action of the representation ρ.

(a) *The case with* Λ_1, $\Lambda_2\neq 0$. In figure 1(a), any point (k_1, k_2) of the lattice can arrive at all the points (k_1', k_2') of the lattice for which $k_1'+k_2'\geqslant k_1+k_2$ under the action of ρ



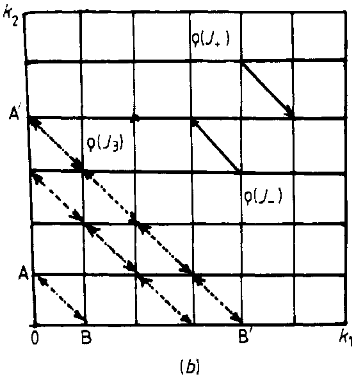


Figure 1. (*a*) $\Lambda_1, \Lambda_2 \neq 0$; (*b*) $\Lambda_1 = \Lambda_2 = 0$.

that forms an invariant subspace $V_{(N)}$: $\{F(k_1', k_2')|k_1' + k_2' \geqslant N\}$ for the fixed $N = k_1 + k_2$ of $\bar{\Omega}/L_{(12)}$. Since there does not exist an invariant complementary subspace for any invariant subspace of $V_{(N)}$, the representation on the space $\bar{\Omega}/L_{(12)}$ is indecomposable. It is easily seen that there is an invariant subspace chain

$$\bar{\Omega}/L_{(12)} = V_{(0)} \supset V_{(1)} \supset V_{(12)} \supset \ldots \supset V_{(N)} \supset V_{(N+1)} \supset \ldots \tag{10}$$

in which each quotient space $V(N, k) = V(N)/V(N+k)$:

$$\{g(k_1, k_2) = F(k_1, k_2) \bmod V(N+k)|N \leqslant k_1 + k_2 \leqslant N + k - 1\} \tag{11}$$

has the finite dimension

$$\dim V_{(N,k)} = \dim V_{(0)}/V(N+k) - \dim V_{(0)}/V(N) = \tfrac{1}{2}(2N+k+1)k. \tag{12}$$

Therefore, the representation defined by (9) induces a finite-dimensional representation with any dimension $\frac{1}{2}(2N+k+1)k$ and the extremal vector $g(N+k-1, 0)$ and $g(0, N+k-1)$ satisfying the relations

$$\rho(J_+)g(N+k-1, 0) = 0 \qquad \rho(J_-)g(0, N+k-1) = 0 \tag{13}$$

on the space $V(N, k)$.

Defining an order for these states $g(k_1, k_2)$

$$g(k_1, k_2) > g(k_1', k_2')$$

$$\Leftrightarrow \begin{cases} k_1 + k_2 > k_1' + k_2' & \text{if } k_1 + k_2 \neq k_1' + k_2' \\ \text{The first non-zero number of the pair} & \\ (k_1 - k_1', k_2 - k_2') \text{ is positive} & \text{if } k_1 + k_2 = k_1' + k_2' \end{cases}$$

we obtain regularly the matrices of the representations on the spaces $V(N, k)$. According to (9), we give two examples as follows.

(i) The three-dimensional representation on the space $V_{(0,2)}$ is

$$\rho(J_+) = \begin{bmatrix} 0 & 1 & \Lambda_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(J_-) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \Lambda_1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(J_3) = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & \Lambda_1 \\ 0 & -1 & -\Lambda_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(ii) The five-dimensional representation on the space $V_{(1,2)}$ is

$$\rho(J_+)=\begin{bmatrix}0&1&0&\Lambda_2&0\\0&0&2&0&\Lambda_2\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\qquad \rho(J_-)=\begin{bmatrix}0&0&0&0&0\\2&0&0&\Lambda_1&0\\0&1&0&0&\Lambda_1\\0&0&0&0&0\\0&0&0&1&0\end{bmatrix}$$

$$\rho(J_3)=\begin{bmatrix}1&0&0&\frac{1}{2}\Lambda_1&0\\0&0&0&-\frac{1}{2}\Lambda_2&\frac{1}{2}\Lambda_1\\0&0&-1&0&-\frac{1}{2}\Lambda_2\\0&0&0&\frac{1}{2}&0\\0&0&0&0&-\frac{1}{2}\end{bmatrix}.$$

(*b*) *The case with* $\Lambda_1=\Lambda_2=0$. In figure 1(*b*), under the action of ρ, the points of the lattice in any straight line A'B' parallel to the straight line AB only move along A'B' and these points form an invariant subspace $V^{[N]}$: $\{F(k_1,k_2)|k_1+k=N\}$ with the extremal vectors $F(N,0)$ and $F(0,N)$ denoted by two ends A and B of the straight line A'B'. Due to the following decomposition:

$$\bar{\Omega}/L_{(1,2)}=\sum_{N=0}^{\infty}\oplus V^{(N)} \tag{14}$$

the representation defined by (9) is completely reducible. On the space $V^{(N)}$, this representation subduces an $(N+1)$-dimensional irreducible representation

$$\begin{aligned}&\rho(J_+)F(k_1,k_2)=k_2F(k_1+1,k_2-1)\\&\rho(J_-)F(k_1,k_2)=k_1F(k_1-1,k_2+1)\\&\rho(J_3)F(k_1,k_2)=\tfrac{1}{2}(k_1-k_2)F(k_1,k_2).\end{aligned} \tag{15}$$

If we choose for the space $V_{(2j)}$ a coupling basis

$$|j,m\rangle=\frac{F(j+m,j-m)}{[(j-m)!(j+m)!]^{1/2}} \tag{16}$$

the representation defined by (9) is rewritten as

$$\begin{aligned}&\rho(J_+)|j,m\rangle=\Lambda_2(j+m+1)^{1/2}|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle+[(j-m)(j+m+1)]^{1/2}|j,m+1\rangle\\&\rho(J_-)|j,m\rangle=\Lambda_1(j-m+1)^{1/2}|j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle+[(j+m)(j-m+1)]^{1/2}|j,m-1\rangle\\&\rho(J_3)|j,m\rangle=\tfrac{1}{2}(\Lambda_1(j+m+1)^{1/2}|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle-\Lambda_2(j-m+1)^{1/2}|j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle)+m|j,m\rangle.\end{aligned} \tag{17}$$

In particular, when $\Lambda_1=\Lambda_2=0$, we obtain the standard angular momentum representation [4] of SO(3):

$$\begin{aligned}&\rho(J_+)|j,m\rangle=[(j-m)(j+m+1)]^{1/2}|j,m+1\rangle\\&\rho(J_-)|j,m\rangle=[(j+m)(j-m+1)]^{1/2}|j,m-1\rangle\\&\rho(J_3)|j,m\rangle=m|j,m\rangle.\end{aligned} \tag{18}$$

There is a need for us to point out that the representations (17) and (18) are not essentially different from (9) and (15). In fact, the representation (18) is just the standard orthogonal form of (15) familiar in angular momentum theory and that, in the same way, (17) could be said to be a standard form of (9) that describes the transformations between the quantum states $|j+\frac{1}{2},m\rangle$ and the quantum states $|j,m'\rangle$.

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